Signal amplification in an agent-based herding model

Adrián Carro*, Raúl Toral, Maxi San Miguel

IFISC (CSIC-UIB),
Campus Universitat de les Illes Balears, E-07122, Palma de Mallorca, Spain

Abstract

A growing part of the behavioral finance literature has addressed some of the stylized facts of financial time series as macroscopic patterns emerging from herding interactions among groups of agents with heterogeneous trading strategies and a limited rationality. We extend a stochastic herding formalism introduced for the modeling of decision making among financial agents, in order to take also into account an external influence. In particular, we study the amplification of an external signal imposed upon the agents by a mechanism of resonance. This signal can be interpreted as an advertising or a public perception in favor or against one of the two possible trading behaviors, thus periodically breaking the symmetry of the system and acting as a continuously varying exogenous shock. The conditions for the ensemble of agents to more accurately follow the periodicity of the signal are studied, finding a maximum in the response of the system for a given range of values of both the noise and the frequency of the input signal.

Keywords: Financial markets, Agent based models, Herding, External information, Resonance
JEL: C63, D89, G14

1. Introduction

The ubiquitous observation of *stylized facts* or non-trivial universal statistical properties (Cont, 2001) in financial time series has traditionally been explained within the efficient market hypothesis as a direct consequence of the statistical properties of the news arrival process. Recent years, however, have witnessed the appearance of a growing number of contributions based on heterogeneous interacting agents (Hommes, 2006) interpreting these stylized facts as the macroscopic outcome of the diversity among the economic actors, and the interplay and connections between them (Kirman, 1992; Lux and Westerhoff, 2009; Farmer and Geanakoplos, 2009; Colander et al., 2009). Heterogeneity refers here to the agents' capability to choose from a set of different market strategies or trading rules. Regarding their interplay, different interaction mechanisms have been studied, whether direct or indirect, global or local.

We will focus on a series of stochastic models of information transmission inspired by Kirman (1991, 1993), and whose main ingredient is their emphasis on the processes of social interaction among agents, based on herding behavior or a tendency to follow the crowd. These

*Corresponding author. Tel.: +34 971 25 98 82; Fax: +34 971 17 32 48
Email address: adrian.carro@ifisc.uib-csic.es (Adrián Carro)

mechanisms of social interplay have also drawn a growing attention from different disciplines during recent decades (Castellano et al., 2009).

Inspired by a series of entomological experiments with ant colonies, Kirman proposed a stochastic herding formalism to model decision making among financial agents (Kirman, 1993). In the experiments with ants, entomologists observed some asymmetric macroscopic patterns emerging from an apparently symmetric situation: when ants were faced with a choice between two identical food sources, a majority of the population tended to exploit only one of them at a given time, switching its foraging attention focus to the other source every once in a while. In order to explain this behavior, Kirman developed a stochastic model where the probability for an ant to change its foraging source is a combination of two terms, one related to pairwise herding interactions or recruitment probability, and the other as an autonomous switching tendency or idiosyncratic behavior. This very simple model can also be interpreted in terms of market behavior by just replacing an ant's binary choice between food sources by a market agent's choice between two trading strategies.

A series of subsequent papers (Lux and Marchesi, 1999; Alfarano et al., 2005, 2008, 2011) has focused on explaining some of the stylized facts observed in empirical data from financial markets in terms of herding models of the Kirman type. Some of the universal features of financial time series which have been reproduced in this framework are: the non-Gaussian or leptokurtic character of the unconditional distribution of returns —also known as "fat tail" shape—, the temporal bursting behavior of the volatility —or volatility clustering—, and the positive autocorrelation of absolute and squared returns.

The main purpose of the present paper is to contribute to the theoretical assessment of the influence of an external significant agent, media group or institution upon a financial market. To this end, we study an agent-based herding model of the Kirman type extended to take into account incoming information. In particular, we add an external periodic signal which will, at regular time intervals, induce a modification of the relative attractiveness of both possible options, thus helping the agents to switch their choice in one direction or the other alternatively. This external forcing can be interpreted, in terms of financial markets, as an advertising or a public perception in favor or against one of the two possible trading behaviors, thus periodically breaking the symmetry of the system. The conditions for the ensemble of agents to more accurately follow the periodicity of the signal are studied. In particular, we measure the response of the system as a function of the tendency of the agents to autonomously switch their trading strategy. We find, for a wide range of the model parameters, a resonance phenomenon, i.e., a maximal amplification of the incoming information as a function of a given parameter of the model. Even if the perfect periodicity of the external signal may not be a realistic feature of real financial markets, it does not affect the generality of our results concerning the conditions for the system to better follow any given signal.

The outline of the paper is as follows. In section 2, we present the original Kirman model, derive the Langevin equation describing the process, find its stationary solution and describe the dynamics in terms of an effective potential. The motivation for a Langevin description is to clearly observe the role played by the parameters of the model, as well as for a direct comparison with the externally forced case, which is analytically developed in section 3. The numerical results obtained with this latter model, their analysis as well as their graphical presentation, are covered in section 4, where we also discuss some particular features of the resonance found. We draw, in section 5, the main conclusions and add some general remarks.

It is also worth mentioning that the method we used to solve the model is different from those used in the previous literature. We applied a Gillespie (1977) algorithm in order to generate

statistically correct trajectories of the stochastic equation. Therefore, we find the unbiased times when the transitions take place. A brief description of this simulation method will be left for a final appendix.

2. The original herding model

A stochastic model of information transmission was proposed by Kirman (1993) in order to explain the collective behavior observed in ant colonies when they faced a choice between two identical food sources located in the neighborhood of their nest. Despite an apparently symmetric situation, being both sources exactly equal and located at the same distance, entomologists observed the systematic emergence of asymmetric macroscopic patterns of foraging among the ants. Indeed, at any point in time, a clear majority of the ant population is found exploiting one of the food sources, while a minority concentrates on the other one[1]. Moreover, the experiments showed that every once in a while a switch occurs in the system and a majority is found foraging in the source previously neglected. Therefore, averaging over time, the probability distribution of the fraction of ants visiting one or the other food sources becomes bimodal.

In order to explain this behavior, Kirman postulates, on the one hand, the existence of a herding propensity among the ants, i.e., a tendency to follow the crowd, which implies the existence of some kind of interaction among them with information transmission. This communication process has been found by entomologists and works mainly through hormone signals or physical recruitment. On the other hand, he also assumes the ants to randomly explore their neighborhood looking for new food sources, so every one of them has an autonomous or idiosyncratic switching probability due to this stochastic search. Thus, the model developed by Kirman basically states that the probability for an ant to change its foraging source is a combination of two terms, one related to pairwise herding interactions or recruitment probability, and the other as an idiosyncratic switching tendency.

This model can be easily interpreted in terms of market behavior and has been used to model contagion and herding phenomena in financial markets in a number of subsequent papers (Alfarano et al., 2005, 2008; Alfarano and Milaković, 2009; Alfarano et al., 2011), some of them including significant variations (Lux and Marchesi, 1999). In a market context, an ant's binary choice between food sources is replaced by a market agent's choice between two trading strategies[2]. For instance, in a foreign exchange market, agents can adopt different trading strategies such as a fundamentalist or a chartist forecast of future exchange rate movements. A further example would be the choice between an optimistic or pessimistic tendency among the chartist traders . In these examples, the Kirman model would be the decision making mechanism among financial agents, who decide whether to buy or sell in a given situation, thus giving rise to market switches between a dominance of one or the other strategy.

Let us now briefly review the formalization of the original Kirman model and the derivation of some analytical results along the lines of previous works by Alfarano et al. (2008) and Kononovicius and Gontis (2012). Let the market, or ant colony, be populated by a fixed number of agents N, and let n be the number of those agents choosing one option, while $N-n$ choose the other one. For the sake of clarity, we will refer hereafter to the agents n as optimistic and to the

[1] So as to make sure that the initial situation is symmetric, the entomologists used also a setting with only one food source and two identical and separated paths, obtaining equivalent results.

[2] These trading strategies may be related to some particular rules for the formation of his expectations about the future evolution of prices, or result from differences in access and interpretation of present and past information.

agents $N-n$ as pessimistic. Thereby, $n \in 0,1,...,N$ defines the configuration of the system. The evolution of the system is given by two terms: on the one hand there are pairwise encounters of agents[3], after which one of them may copy the strategy of the other, and on the other hand there are idiosyncratic random changes of state.

There have been mainly two different implementations of the pairwise encounter term in the literature. In his seminal 1993 paper, Kirman proposed the use of a herding intensity that, for each agent, is proportional to the fraction of agents in the opposite state. One of the main drawbacks of this original formalization has been pointed out to be its lack of robustness with respect to an enlargement of the system size, or N-dependence, since an increasing number of participants in the market causes the stochasticity to vanish and therefore the stylized facts to fade away[4]. On the contrary, some later authors (Alfarano et al., 2005, 2008; Alfarano and Milaković, 2009; Alfarano et al., 2011) avoided this problem with an alternative modelization of the pairwise interaction mechanism with a herding intensity that, for each agent, is proportional to the total number of agents in the opposite state, thus allowing each individual to interact with any other regardless of the system size. We will hereafter adopt the second and more recent formalism.

An additional assumption of the model is the lack of memory of the agents, so their probability of changing state does not depend on the outcome of previous encounters, neither on previous idiosyncratic switches. Therefore, the stochastic evolution of the system can be formalized as a Markov process depending, at each time step, just on the probability to switch from the present configuration of the system n to some other state n' in a time interval Δt, denoted by $P(n',t+\Delta t|n,t)$. However, if this time interval Δt is taken to be small enough, then the probability to observe multiple jumps is negligible and we can restrict our analysis to $n' = n \pm 1$. Furthermore, the probabilities would then be related to the transition rates per unit time as $P(n',t+\Delta t|n,t) = \pi(n \to n')\Delta t$. According to the evolution of the model described above, the transition rates for each individual i, $\pi_i^+ = \pi_i(\textit{pessimistic} \to \textit{optimistic})$ and $\pi_i^- = \pi_i(\textit{optimistic} \to \textit{pessimistic})$, can be formally defined as

$$\pi_i^+ = a + hn,$$
$$\pi_i^- = a + h(N-n),$$

where the parameters a and h stand for the idiosyncratic switch and the herding intensity coefficients respectively[5]. The rates for the whole system are, therefore,

$$\begin{aligned} \pi^+(n) = \quad & \pi(n \to n+1) = \quad (N-n)\,(a+hn), \\ \pi^-(n) = \quad & \pi(n \to n-1) = \quad n\,(a+h(N-n)). \end{aligned} \tag{1}$$

There are two parameters in the model, a and h, but one of them can be used as a rescaling of the time variable, so that there is only one relevant parameter, such as $\varepsilon = \frac{a}{h}$.

[3]Having one-to-one meetings implicitly assumes the density of agents to be low enough so as to avoid multi-agent meetings.

[4]By implementing the dynamics in different network structures and hierarchies, Alfarano and Milaković (2009); Alfarano et al. (2011) pointed out that the problem of N-dependence is related to the structural heterogeneity of the model.

[5]Obviously, only one of these transition rates would apply at each time step for each agent i: π_i^+ in case agent i is currently in the pessimistic or -1 state, and π_i^- if it is currently in the optimistic or $+1$ state.

Using the one-step increment and decrement operators[6], $\mathbf{E}$ and $\mathbf{E}^{-1}$ (Van Kampen, 2007), and calling $P(n,t)$ the probability to have n agents in the optimistic state at time t, the master equation describing the process is written as

$$\frac{\partial P(n,t)}{\partial t} = (\mathbf{E}-1)\left[\pi^-(n)P(n,t)\right] + (\mathbf{E}^{-1}-1)\left[\pi^+(n)P(n,t)\right]. \tag{2}$$

We now change from the extensive variable n in the range $n \in [0,N]$ to an intensive variable $m = 2\frac{n}{N} - 1$, giving the population configuration or opinion index in the range $m \in [-1,+1]$. This intensive variable m can be thought of as continuous if the number of agents N is large enough. Note, however, that the limit of an infinite number of agents is never the case in real social and economical systems, where finite size effects may play a role (Toral and Tessone, 2007). Note as well that a population configuration $m = 0$ would imply a perfect balance of opinions, while $m = -1$ and $m = +1$ would signal a total agreement on the pessimistic and the optimistic opinions respectively. For this new variable, the partial derivation is $\frac{\partial}{\partial n} = \left(\frac{2}{N}\right)\frac{\partial}{\partial m}$ and the relation between the old and the new variable probabilities becomes $P(n,t) = P(m,t)\frac{2}{N}$. If we also take into account that the one-step increment operator can be expressed as $\mathbf{E} = \exp\left\{\frac{\partial}{\partial n}\right\}$ and, furthermore, we assume a large number of market participants N, then this one-step increment operator term can be approximated by a Taylor series up to second order in $\frac{1}{N}$ as

$$(\mathbf{E}^{\pm}-1) = \left(e^{\pm\frac{2}{N}\frac{\partial}{\partial m}} - 1\right) = \left[\pm\frac{2}{N}\frac{\partial}{\partial m} + \frac{2}{N^2}\frac{\partial^2}{\partial m^2}\right] + O\left(\frac{1}{N^3}\right).$$

Introducing this and the transition rates (1) into the expression (2), the master equation can be approximated by a Fokker-Planck equation,

$$\frac{\partial P(m,t)}{\partial t} = \frac{\partial}{\partial m}\left[2amP(m,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial m^2}\left[\left(\frac{4a}{N} + 2h(1-m^2)\right)P(m,t)\right], \tag{3}$$

where $\mu(m) = -2am$ plays the role of a drift term and $D(m) = \frac{4a}{N} + 2h(1-m^2)$ is the diffusion coefficient.

The steady state solution of the Fokker-Planck equation (3) is

$$P_{\mathrm{st}}(m) = \mathscr{Z}^{-1}\left[\frac{a}{2Nh} + \frac{(1-m^2)}{4}\right]^{\frac{a}{h}-1}, \tag{4}$$

where $\mathscr{Z}^{-1}$ is a normalization factor. Observing the functional form of this solution, one can notice that the sign of the exponent will determine whether the probability distribution is unimodal with a peak centered at $m = 0$ or bimodal with peaks at the extremal values $m = -1$ and $m = +1$. Therefore, when the idiosyncratic switching a is larger than the herding intensity h, we find a unimodal distribution, meaning that, at any point in time, the most likely outcome of an observation is to find the community of agents equally split between both options. On the contrary, when the herding h exceeds the idiosyncratic switching intensity a, a bimodal distribution is found, meaning that, at any point in time, the most likely outcome of a static observation is to

[6] The increment operator $\mathbf{E}$ is defined as $\mathbf{E}[f(n)] = f(n+1)$, where $f(n)$ is an arbitrary function of the integer argument n, whereas the decrement operator $\mathbf{E}^{-1}$ has the effect $\mathbf{E}^{-1}[f(n)] = f(n-1)$.

find a large majority of agents choosing the same option. Nevertheless, in different observations, the option chosen by the majority may be different. Note as well that when $a = h$ the probability distribution is uniform. Because of the ergodicity of the model, these probability distributions can also be understood in terms of the fractional time spent by the system in each state.

For a large number of agents N, we can neglect the first term inside the brackets in the steady state solution (4), and thereby write the normalization factor $\mathscr{Z}^{-1}$ as[7]

$$\mathscr{Z}^{-1} = \frac{\Gamma(2a/h)}{\Gamma(a/h)^2} = \frac{\Gamma(2\varepsilon)}{\Gamma(\varepsilon)^2}.$$

Returning to the Fokker-Planck equation (3) and applying the usual transformation rule (Van Kampen, 2007), within the Itô (1951) convention, we find the Langevin or stochastic differential equation describing the process,

$$\begin{aligned} \dot{m} &= \mu(m) + \sqrt{D(m)} \cdot \xi(t) \\ &= -2am + \sqrt{\frac{4a}{N} + 2h(1-m^2)} \cdot \xi(t), \end{aligned} \tag{5}$$

where $\xi(t)$ is a Gaussian white noise, i.e., a random variable with a zero mean Gaussian distribution, $\langle \xi(t) \rangle = 0$, and no autocorrelation, $\langle \xi(t)\xi(t') \rangle = \delta(t-t')$.

Let us first analyze the role played by the noise or diffusive function inside the square root, $D(m) = \frac{4a}{N} + 2h(1-m^2)$. The first term, dependent on a and inversely proportional to the system size N, is related to the "granularity" of the system, and so it vanishes in the continuous limit $N \to \infty$. It basically states that for any finite system there are always finite-size stochastic fluctuations related to the fact that the agents have the ability to randomly change their choice. The second term of the diffusion function is a multiplicative noise term, i.e., a noise whose intensity depends on the state variable itself. Furthermore, it is the only term dependent on the herding coefficient, so we will refer to it hereafter as herding term. As this multiplicative noise is maximum for $m = 0$ and vanishes for $m = \pm 1$, it tends to move the system away from the center and towards those extremes by making a random partial agreement on one or the other possible opinions grow to a complete consensus. Note that for $a = 0$ the consensual states $m = \pm 1$ become absorbing states. We will nonetheless refer to them as absorbing states also for $a \neq 0$, even if they are not properly absorbing in that case.

Regarding the deterministic drift term, $\mu(m) = -2am$, it drives the system back to a balanced position at the center of the opinion index, $m = 0$. Therefore, we have a competition between two driving forces. One of them is of a stochastic nature, it is dominated by the herding term

[7]Certainly we can also apply the probability normalization condition without neglecting the N-dependent term. In this last case, the normalization factor $\mathscr{Z}^{-1}$ would be

$$\mathscr{Z}^{-1} = \frac{\frac{1}{2}\left(\frac{2\varepsilon}{N}+1\right)^{-\varepsilon}\sqrt{\frac{2\varepsilon}{N}+1}}{B\left[\frac{1}{2}\left(1+\sqrt{\frac{2\varepsilon}{N}+1}\right);\varepsilon,\varepsilon\right] - B\left[\frac{1}{2}\left(1-\sqrt{\frac{2\varepsilon}{N}+1}\right);\varepsilon,\varepsilon\right]},$$

where $B[x;a,b]$ is the incomplete beta function, given by

$$B[x;a,b] = \int_0^x u^{a-1}(1-u)^{b-1}du.$$

for large N and it tends to favor the formation of a majority of agents sharing the same opinion. Whereas the other is of a deterministic nature, it is related to the idiosyncratic switches and it tends to break these majorities and drive the system back to a balanced situation, where the agents are equally distributed between both opinions. Depending on the relative magnitude of the idiosyncratic and the herding parameters, a and h, one or the other behavior prevails. Indeed, the particular functional form of the noise in the Kirman model induces a transition in the dynamics of the system, from a monostable to a bistable behavior when increasing the value of h relative to that of a.

In order to clearly observe this noise-induced transition, let us directly write here an effective potential for the Fokker-Planck equation (3) which takes into account the effect of the multiplicative noise over the deterministic potential related to the drift term,

$$V_{\text{eff}}(m) = (h-a)\ln(1-m^2). \tag{6}$$

The derivation of this effective potential, basically consisting of assuming an exponential functional form for the stationary probability distribution, is left for Appendix B. It is obvious, from equation (6) and Figure 1, that a transition will occur at $h = a$ from a one well to a double well potential, for increasing h.

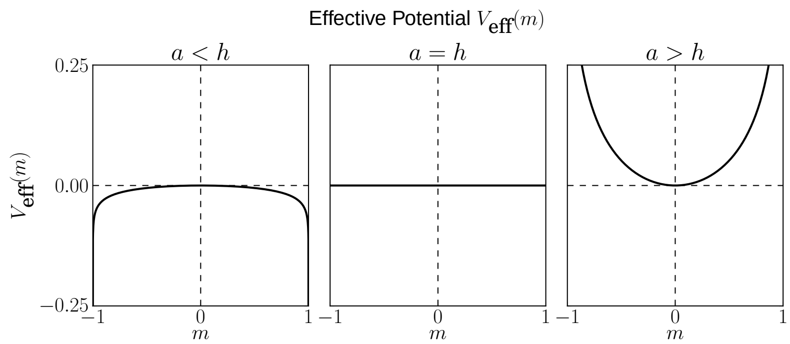

Figure 1: Effective potential $V_{\text{eff}}(m)$ for a herding intensity $h_0 = 0.01$ and three different values of the idiosyncratic switching tendency, $a = 10^{-3}$ ($a < h$), $a = 10^{-2}$ ($a = h$) and $a = 10^{-1}$ ($a > h$).

Another interesting feature worthy of remark here is the fact that, although both wells at the extremes of the opinion space in the effective potential are theoretically infinite, in fact, they are not absorbing states and so the system will leave them with rate proportional to a. This can be understood by a reexamination of the Langevin equation (5), where we notice that precisely at the extremal states $m \pm 1$ the only term acting upon the system is the deterministic force towards the center of the opinion space. To sum up, the role of the herding h is to induce a bistable effective potential, while that of the idiosyncratic switching a is to allow for transitions between both extremal wells.

An example of a stochastic realization of equation (5) is shown in the panel **a** of Figure 3 for a herding intensity ten times larger than the idiosyncratic switching coefficient, $\varepsilon = \frac{a}{h} = \frac{1}{10}$. We can observe, in this panel, the tendency of the system to be temporarily absorbed around the consensual states with random switches between them. Some of those switches are not successful and the system returns to the previous consensus before reaching the opposite one.

3. The model with an external signal

We now proceed to extend the model in order to account for an input of external information able to influence the choices made by the economic actors. We are concerned here with the search for the conditions which allow the market to more accurately follow an information arrival process. To this end, we basically add a weak periodic input signal or driving force, which will, at regular time intervals, induce a modification of the transition rates helping the switches in one or the other direction alternatively. The only reason for the signal to be perfectly periodic is to simplify the measurement and quantification of the similarity between the input driving signal and the output results of the model in terms of market behavior. Nevertheless, our results concerning the search for the parameter ranges for the market to more accurately follow the external signal are general and independent of its particular shape. The only relevant features of the driving signal which have a significant effect on the results are its strength and its frequency or rate of change. In the context of a financial market, this external signal can be interpreted as an advertising, a fad or a public perception in favor or against one of the two possible trading behaviors, thus periodically breaking the symmetry of the system.

We are interested in the modification of the social processes of opinion formation and propagation of information among the economic agents. So we are naturally led to introduce the external forcing in the social term of the transition rates, that is, in the herding coefficient h. Thus, we modify the transition rates (1) as

$$\begin{aligned} \pi^+(n,t) = \quad & \pi(n \to n+1,t) = \quad (N-n)\,(a+h_+(t)n)\,, \\ \pi^-(n,t) = \quad & \pi(n \to n-1,t) = \quad n\,(a+h_-(t)(N-n))\,, \end{aligned} \tag{7}$$

where the herding coefficients are now different in the two possible directions and both are time dependent functions given by

$$\begin{aligned} h_+(t) = \quad & h_0 + \frac{F}{N}\cos(\Omega t), \\ h_-(t) = \quad & h_0 - \frac{F}{N}\cos(\Omega t), \end{aligned} \tag{8}$$

with h_0 playing the role of an average herding coefficient, F acting as the force or signal intensity applied to the whole system and Ω as the signal frequency. The reason for adding and subtracting the cosine term, respectively in $h_+(t)$ and $h_-(t)$, is just so that the herding coefficients are in anti-phase. Note that the forcing term is not proportional to the total force exerted on the system, but to the total force per agent, $\frac{F}{N}$. The rationale behind this particular functional form of the signal intensity is basically a limited resources assumption: the resources spent in order to exert the force on the whole system are divided among its constituents, so that if the system size N increases, the resources spent on each of the agents decrease. In order to keep the transition rates always positive, this intensity must satisfy $\frac{F}{N} \leq h_0$.

Proceeding in a similar manner as for the original model in section 2, and applying the same approximations, we find the new Fokker-Planck equation:

$$\begin{aligned} \frac{\partial P(m,t)}{\partial t} = & \frac{\partial}{\partial m}\left[\Big(2am - F(1-m^2)\cos(\Omega t)\Big)P(m,t)\right] \\ & + \frac{1}{2}\frac{\partial^2}{\partial m^2}\left[\left(\frac{4a}{N} + 2h_0(1-m^2)\right)P(m,t)\right]. \end{aligned} \tag{9}$$

Note that, compared to the previous Fokker-Planck equation (3), the herding coefficient h has been replaced by its average or constant part, h_0. Furthermore, there is a new time dependent term inside the drift function, $\mu(m,t) = -2am + F(1-m^2)\cos(\Omega t)$. Again, the conventional transformation rule leads us, within the Ito form, to the Langevin equation describing the process,

$$\begin{aligned} \dot{m} &= \mu(m,t) + \sqrt{D(m)} \cdot \xi(t) \\ &= -2am + F(1-m^2)\cos(\Omega t) + \sqrt{\frac{4a}{N} + 2h_0(1-m^2)} \cdot \xi(t), \end{aligned} \tag{10}$$

where $\xi(t)$ is, as before, a Gaussian white noise.

The constant part of the herding coefficient, h_0, plays exactly the same role as the herding coefficient itself in the original Kirman model. The new term of the drift function, on the contrary, changes fundamentally the general behavior of the system. In particular, it will try to force the system to follow the periodicity of the signal, as it will periodically favor or oppose the tendency towards $m = 0$ caused by the first drift term. The new parameter F, the strength of the signal, modulates the intensity of this effect. Note also that the factor $(1-m^2)$ inside the new drift term causes its effects to vanish at the extremes of the population configuration space and its absolute value to be maximal at its center for $\cos(\Omega t) = \pm 1$. As a consequence, this new term does not help the system to exit the absorbing states, and thus some idiosyncratic behavior ($a > 0$) is still needed in order to observe transitions between both full agreement states. On the other hand, the equilibrium point of the drift term is no longer a constant at $m = 0$, but a periodic function of time around this central point.

Concerning the competition between the deterministic and the stochastic terms of equation (5), the inclusion of an additional external signal in (10) has the general effect of periodically counteracting or enhancing the deterministic force. For a deeper understanding of the transition induced by the multiplicative noise upon the deterministic potential and the symmetry breaking role of the external signal, let us write an effective potential for the Fokker-Plank equation (9),

$$V_{\text{eff}}(m,t) = (h_0 - a)\ln(1-m^2) - mF\cos(\Omega t). \tag{11}$$

Again, we leave its derivation for Appendix B. Note that the new term, related with the input signal, is linear in the magnetization variable m, and thereby it breaks the symmetry ($m \leftrightarrow -m$) of the potential, for $\cos(\Omega t) \neq 0$.

The time evolution of the effective potential is shown in Figure 2, where five different time snapshots are presented for three values of the parameter a. For sufficiently small values of a the system is not be able to follow the signal, even if a double well effective potential has been induced, since these values are too small to allow for the system to exit the wells. Larger values of a but still smaller than h_0 allow both for the creation of a double well potential and transitions between them, while intermediate values of a larger than h_0 give rise to a rather wide monostable effective potential where the system can still be largely driven by the external signal. Very large values of a lead to narrow effective potentials with minima moving closely around the center of the opinion space.

The particular modification of the behavior of the system due to the introduction of the forcing signal depends on the specific values of the model parameters. Figure 3 contains four panels which show the effect of increasing the intensity of the signal from $F = 0$ to its maximum allowed value $F = Nh_0$. Note that the values chosen for the idiosyncratic switching and the herding parameters are, respectively, $a = 0.001$ and $h_0 = 0.01$, so we find a fairly bimodal phase, as in

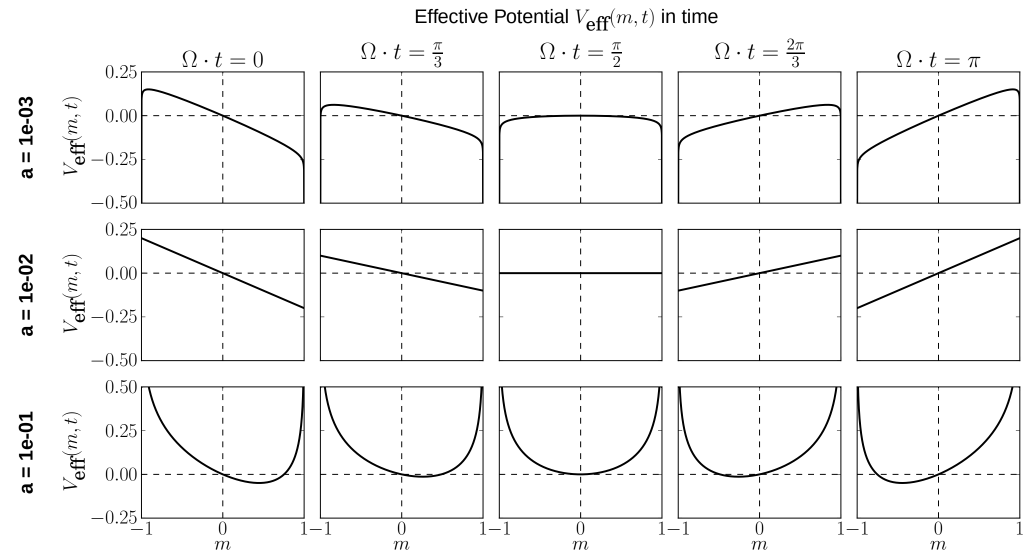

Figure 2: Snapshots of the effective potential $V_{\text{eff}}(m,t)$ in various time instants $t = t^*$ and for three different values of the idiosyncratic switching tendency, $a = 10^{-3}$, 10^{-2}, 10^{-1}. The rest of the parameter values are $h_0 = 0.01$, $F = 0.2$ and $\Omega = 0.02$. Note that the values of a shown here correspond to the three main cases $a < h_0$, $a = h_0$ and $a > h_0$.

the unforced case. We observe the important differences between both upper panels (**a** and **b**) and those below them (**c** and **d**): the system does not follow the periodicity of the input signal in the former cases, while it certainly and quite accurately does in the latter ones. Obviously, no periodic behavior is expected in panel **a**, where there is no input signal ($F = 0$). Regarding panel **b**, characterized by a weak signal intensity, we observe a behavior quite similar to the previous unforced case, with no appreciable periodicity. Nonetheless, the particular trajectory of the system is clearly different and we notice a certain increase in the number of transitions between both extremal states, which point at the fact that a weak signal intensity is already able to facilitate some transitions. Even if in both panels **c** and **d** the system shows a clear order in time, it is also noticeable that in the first case, panel **c**, the periodicity is not so accurately followed as in the second one, panel **d**: some periods are longer or shorter than the others, some transitions are missing, there is a longer phase lag, etc.

4. Resonance phenomena

We present, in this section, the main results obtained from the analysis of the model introduced in section 3 concerning the existence of a resonance phenomenon, i.e., a maximum in the response of the system to the external signal as a function of a given control parameter. As we have already pointed out, the capacity of the system to follow the input signal is basically related to the presence of some idiosyncratic behavior or random autonomous decision making in the system ($a > 0$). Therefore, we choose this idiosyncratic switching intensity a to be our control parameter, while we keep constant the average herding coefficient h_0 and the forcing strength F. In particular, we fix their values as $h_0 = 0.01$ and $F = 0.2$, as in panel **c** of Figure 3. Finally, we have chosen to show here the results for a number of agents $N = 200$, although we

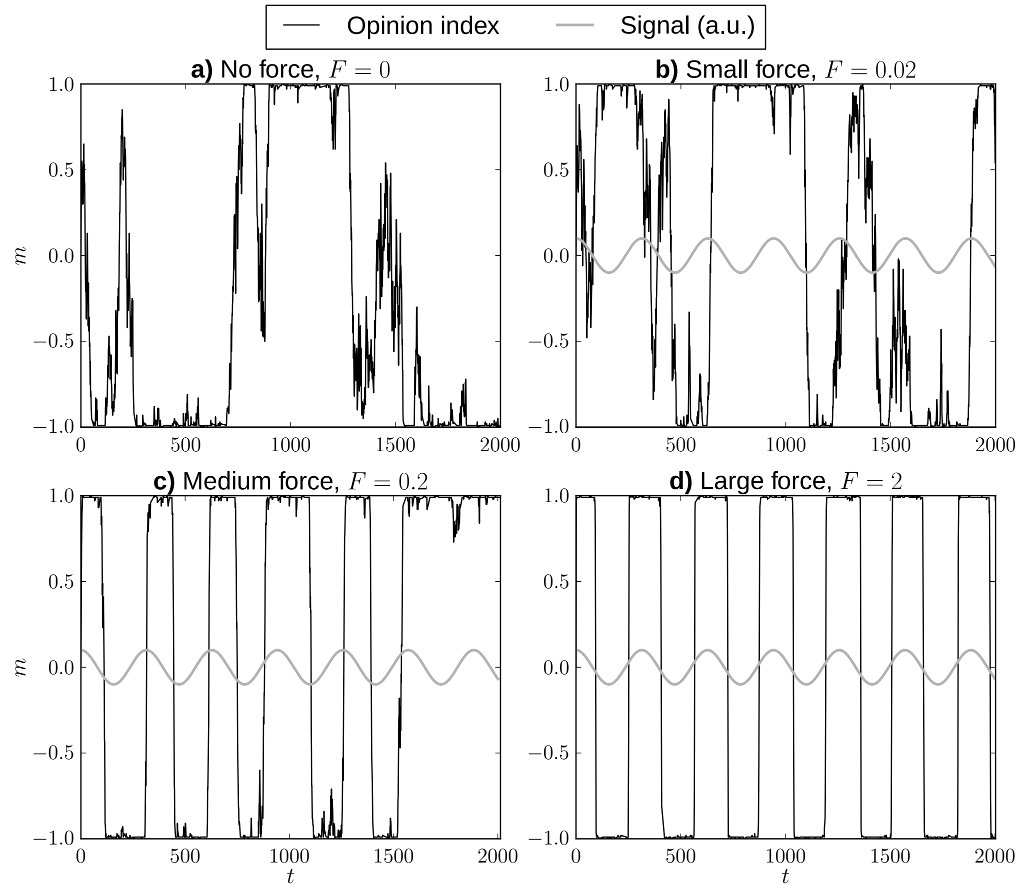

Figure 3: Black line: Time evolution of the opinion index simulated for different values of the signal intensity F and fixed values of the parameters $a = 0.001$, $h_0 = 0.01$, $\Omega = 0.02$ and $N = 200$. Grey line: External periodic signal in arbitrary units (a.u.), intended only for period comparison.

also performed simulations with $N = 50$ and $N = 800$, finding a generally equivalent behavior as discussed below. The signal intensity per agent, $\frac{F}{N}$, is thus only a 10% of the herding coefficient.

Let us start by considering the influence that varying the idiosyncratic switching tendency a has over the trajectories or time series of the opinion index m (see Figure 4). Note that there is no maximum allowed value for this variable, as it was the case with the signal intensity F: its only constraint is that it must be $a > 0$. Therefore, we simply choose a reasonable range which includes the different behaviors described in the previous section and observed in Figure 2: from a fully bimodal case, with almost two deltas at the extremes of the probability distribution of states; up to a fully unimodal case, with an almost perfect Gaussian distribution.

The first of these cases (panel **a**) corresponds to a value $a = 0.0001 = \frac{h_0}{100}$. It is clear that, in this case, the system does not follow the periodicity of the input signal and, in fact, it stays pretty close to a full agreement state for most of the time. This is due to the extremely low level of idiosyncratic behavior compared to the herding intensity, which makes it very difficult

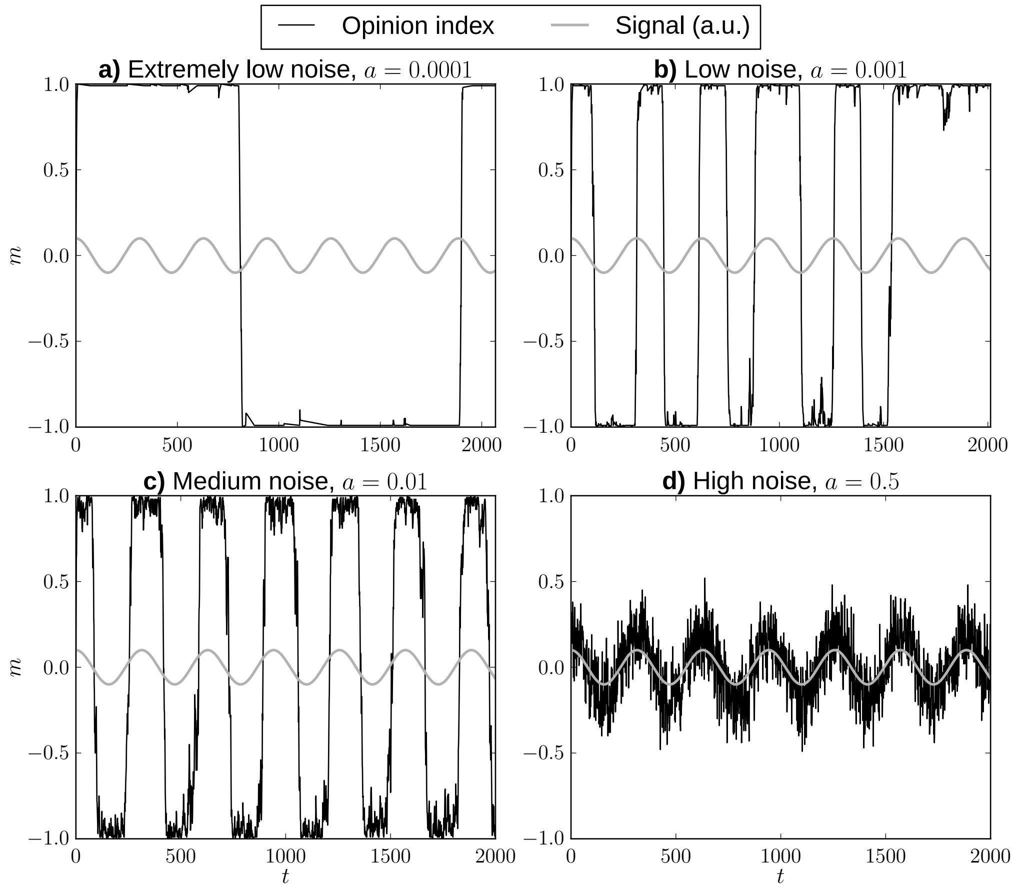

Figure 4: Black line: Time evolution of the opinion index simulated for different values of the idiosyncratic switching tendency a and fixed values of the parameters $h_0 = 0.01$, $F = 0.2$, $\Omega = 0.02$ and $N = 200$. Grey line: External periodic signal in arbitrary units (a.u.), intended only for period comparison.

for the system to exit these consensual states and, therefore, the transitions between both of them cannot be as frequent as the forcing tries to impose. Even if the example shown in panel **b**, corresponding to a value $a = 0.001 = \frac{h_0}{10}$, was as well in the bimodal region in the original model, the idiosyncratic behavior intensity is already strong enough so as to take the system out of the absorbing states with a frequency allowing it to fit the external signal in almost all the transitions. Note, for instance, that the last switch is missing in the trajectory. This is not the case in panel **c**, where the idiosyncratic coefficient equals the herding intensity, $a = 0.01 = h_0$, and thus the probability distribution of states is uniform in the unforced model. We observe here that the system easily follows the periodicity of the input signal, although the trajectory starts to be fairly noisy. The case shown in panel **d**, with $a = 0.5 = 50 \cdot h_0$, corresponds to a rather unimodal probability distribution of states in the original model. The output trajectory accurately follows the period of the forcing signal. However, the great strength of the first term of the drift function, the one proportional to a (see equation (10)), prevents the system from

collapsing to the consensual states. Therefore, the competition between the first and the second terms of the drift function keeps the system oscillating in the proximity of the balanced situation, $m = 0$.

We note that, by applying a driving signal whose absolute strength per agent is only a 10% of the herding intensity, we can obtain (for instance in panel **c** of Figure 4) a large amplification of this input, as the system tends to collapse to the extremal states $m = \pm 1$. This is because, on the one hand, the driving signal breaks locally in time the symmetry of the problem, by favoring periodically one or the other states, and on the other hand, the stochastic or diffusive term (see equation (10)) tends to make any asymmetry grow towards a total consensus, as has been reported above. However, this effect is not observed for every value of the idiosyncratic switching parameter a: it is clearly not present in panels **a**, where there is even no periodic behavior, and **d**, where there is no collapse to the extremes. In short, what we observe is a maximum in the amplification of an external driving signal as a function of the intensity of the agents' idiosyncratic behavior, a phenomenon which correctly fits the definition of resonance.

In order to quantify this resonance phenomenon, we consider the quality of the global response following the input forcing, that is: how much the system output m fits the input signal $\frac{F}{N}\cos(\Omega t)$. For this purpose, we use the *spectral power amplification factor* η (Gammaitoni et al., 1998),

$$\eta = \frac{4}{F^2} |\langle e^{-i\Omega t} m(t) \rangle|^2, \tag{12}$$

which is basically the normalized amplitude of the Fourier component at the forcing frequency Ω of the time series $m(t)$. Note that the signal intensity used for the normalization is the absolute value of the total force applied to the whole system, F. A larger η is related with a better entrainment of the system by the external signal. Note that a larger entrainment can be caused both by a better fit of the input signal or by its amplification.

The results obtained for the spectral power amplification factor η are shown in Figure 5 for three different system sizes. The same general behavior is observed in all of them: there is a region in the parameter space in which the system response to the weak periodic driving signal displays a maximum value as a function of the idiosyncratic switching intensity a. Furthermore, the particular values of these maxima are quite large, all of them implying a large amplification of the input signal. Note as well that this amplification occurs for a fairly wide range of values of the control parameter a in all three cases. These results suggest that there is a certain range of idiosyncratic behavior intensity which largely improves the entrainment of the system by an external periodic signal.

The only difference worthy of remark between the three examples in Figure 5 is due to the granularity term, the only one dependent on N. In fact, the three sizes collapse in the same curve for higher values of a, while they are clearly different for intermediate values and join again together for the lower a region. This behavior is caused by the unequal importance of the two a dependent terms in the Langevin equation (10): the granularity and the first drift term. For very small values of the control parameter a, the system stays most of the time absorbed in the consensual states, so neither of these terms has any noticeable importance. In the intermediate a region, the granular N dependent term plays a relevant role in taking the system out of the consensual states with a probability uncorrelated with the frequency of the input signal. Therefore, a large granularity term —small number of agents N— leads to a worse amplification of the input signal and, thereby, to a lower spectral power amplification factor η. Finally, for large values of the control parameter, the system does not even reach the absorbing states and the behavior is predominantly led by the first term of the drift function, so the granular effects become negligi-

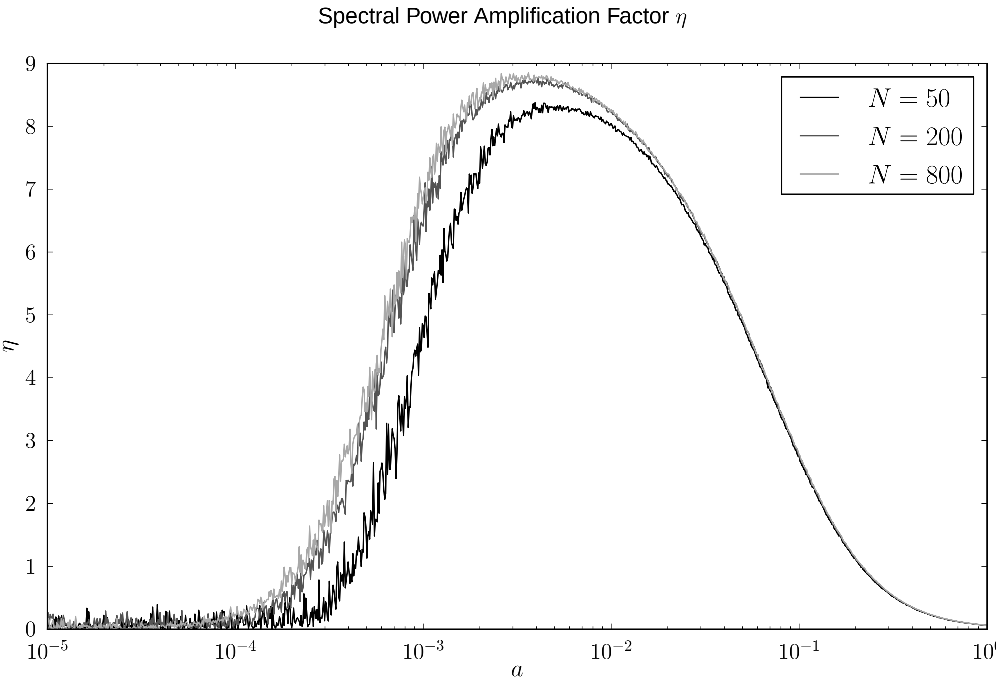

Figure 5: *Spectral power amplification factor* η, from equation (12), as a function of the idiosyncratic switching tendency a and with the parameter values $h_0 = 0.01$, $F = 0.2$ and $\Omega = 0.02$. Results are shown for three different system sizes, $N = 50,\ 200,\ 800$.

ble. Note as well that the relative difference between the curves is much smaller when comparing the $N = 800$ with the $N = 200$ cases than when comparing this latter with the $N = 50$ example. This is simply a consequence of the granularity term being inversely proportional to the system size N, so for larger and larger N the results become more and more similar.

Let us now discuss some insights regarding the particular features of the resonance mechanism found. Indeed, this mechanism is reminiscent of a phenomenon known as *stochastic resonance* (Benzi et al., 1981, 1982; Nicolis and Nicolis, 1981), since the maximum in the response of the system to the external signal is related to the relative importance of the stochastic term as compared to the deterministic one. Note, however, that in our case (see equation (10)) this is only true for N tending to infinity. Furthermore, if in a stochastic resonance the control parameter would normally be the intensity of the stochasticity (related to h in our case), we have chosen the intensity of the deterministic term, a, as our control parameter. Both the stochastic resonance and the one here presented take place when a deterministic potential is driven by a combination of a periodic and a random forcing. Nevertheless, the roles played by this random force and the deterministic potential are fundamentally different in our case: both factors exchange their roles. In the case most commonly found in the literature (Gammaitoni et al., 1998), the potential has a double-well shape and the noise intensity or diffusion function is additive and is a constant for the whole variable range. On the contrary, the case here presented is characterized by a monostable deterministic potential and a multiplicative noise intensity which is maximum at the center of the opinion space and vanishes at its extremes. Therefore, the deterministic force pulls the

system towards the center while the net effect of the random forces is to drive the system out of the center and towards the extremes, in contrast to the usual stochastic resonance scenario.

5. Conclusions

The aim of the present paper is to advance towards a quantitative understanding of the influence of an external source of information upon a financial market characterized by a certain herding behavior. The stochastic formalism used as a point of departure for our investigation incorporates individual behavioral heterogeneity as well as a tendency for social interaction, in the tradition of Kirman's seminal ant colony model. We have extended this original model by adding a periodical signal modeling a varying incoming information from an external source.

A transition takes place in the original Kirman model from a monostable to a bistable behavior when increasing the herding propensity of the agents with respect to their idiosyncratic tendency. We have reinterpreted this noise-induced transition in terms of the mono- or bistability of an effective potential. The monostable case can be understood as a market where each of both possible strategies is used by approximately half of the traders, while the bistable configuration corresponds to a market where there is always a clear majority of traders using a given strategy, even if the chosen one can change over time. In this context, we have shown that, when an input of external information is taken into account, a resonance phenomenon may take place under certain circumstances. In particular, we have found a certain range of values of the idiosyncratic behavior intensity a that maximizes the response of the system to a weak periodic input signal. The specific range of values for which an amplification is observed depends on the intensity and the frequency or rate of change of this incoming information.

Within a financial market framework like the one presented by Alfarano et al. (2008), a maximal amplification of the external signal would simply mean that the incoming information influences virtually the whole market and is able to convince most of the traders to a particular mood, to act in a certain manner or to use a specific trading strategy, whether it is an optimistic or pessimistic mood, a buying or selling activity, a chartist or fundamentalist strategy, etc. We have shown the existence of three different regimes regarding the capability of the system to follow the external signal. In financial terms, we could translate these regimes as:

1. Impossibility for the market to assimilate the incoming information in a reasonable time, that is, in a time scale shorter than that of the variations of the information itself (values of the idiosyncratic switching intensity a below the range of the resonance).
2. Exaggeration of incoming shocks of information, i.e., even small outside changes lead to large market movements (values of a within the range of the resonance).
3. Precise assimilation of incoming information but with a very small effect in the market (values of a above the range of the resonance).

The amplification of small external signals can lead to enormous price variations, a large volatility and, therefore, to a great instability of the market. The results presented in this paper support the idea that the greater intensity of the herding with respect to the idiosyncratic tendency may play an important role in the development of such instabilities in a financial market open to external incoming information.

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Appendix A. Simulation method

In contrast with the methods presented in the previous literature, we have used a Gillespie algorithm for the simulation of the model (Gillespie, 1977). A single realization of this algorithm represents a random walk with the exact probability distribution of the master equation, therefore generating statistically correct trajectories of the stochastic equation. In particular, we find the unbiased times when the transitions take place.

For a stochastic system with time dependent transition rates[8] $\pi^{\pm}(t) = \pi(n \to n \pm 1, t)$, and given that the last transition took place at time t_1, the cumulative probability of observing an event $n \to n \pm 1$ at time t_2 can be written as

$$F^{\pm}(t_2|t_1) = 1 - e^{-\int_{t_1}^{t_2} \pi^{\pm}(t)dt},$$

and we can equate this expression to a uniformly distributed random number $u^{\pm}$ between 0 and 1 in order to find an equation for the time of the next event, t_2,

$$\int_{t_1}^{t_2} \pi^{\pm}(t)dt = -\ln(1-u^{\pm}) \equiv -\ln(u^{\pm}),$$

where we have used that $u^{\pm}$ and $1-u^{\pm}$ are statistically equivalent. By that means, two different times are found: t_2^+, corresponding to the transition rate $\pi^+(t)$, and t_2^-, related to the transition rate $\pi^-(t)$. The event actually taking place is the one with a shortest time.

Applying the last result to the transition rates of the original Kirman model, as given in equation (1), and solving for t_2 we find

$$t_2^+ = t_1 - \frac{\ln(u^+)}{(N-n)\,(a+hn)},$$

$$t_2^- = t_1 - \frac{\ln(u^-)}{n\,(a+h(N-n))}.$$

Regarding the forced case, whose transition rates are given by (7) and (8), we find

$$t_2^+ = t_1 - \frac{\ln(u^+)}{(N-n)\,(a+h_0 n)} + \frac{F}{N\Omega}\,\frac{n}{a+h_0 n}\left(\sin(\Omega t_1) - \sin(\Omega t_2^+)\right),$$

$$t_2^- = t_1 - \frac{\ln(u^-)}{n\,(a+h_0(N-n))} + \frac{F}{N\Omega}\,\frac{(N-n)}{a+h_0(N-n)}\left(\sin(\Omega t_2^-) - \sin(\Omega t_1)\right),$$

equations whose solution can be approximated by different numerical methods.

[8]These rates may also depend on other variables, as it is the case in the Kirman model, but we are here interested only in the time dependence.

Appendix B. Effective potential

We derive, in this appendix, the effective potential both for the original and the forced Kirman dynamics, presented in the equations (6) and (11). Let us start by defining the "effective potential" $V_{\text{eff}}(m)$ as

$$P_{\text{st}}(m) \equiv \mathscr{Z}^{-1} \exp\left(-\frac{V_{\text{eff}}(m)}{D}\right),$$

where $P_{\text{st}}(m)$ is the stationary state probability distribution, D is an effective noise intensity and the constant $\mathscr{Z}^{-1}$ plays the role of a normalization factor.

For the general Fokker-Planck equation

$$\frac{\partial P_{\text{st}}(m,t)}{\partial t} = -\frac{\partial}{\partial m}\left[q(m)P(m,t)\right] + \frac{\partial^2}{\partial m^2}\left[Dg(m)P(m,t)\right],$$

the stationary distribution is found by assuming $\frac{\partial P_{\text{st}}}{\partial t} = 0$ and solving the resulting equation. By this means, the general effective potential can be written as

$$V_{\text{eff}}(m) = -\int \frac{q(m)}{g(m)^2} dm + 2\int \frac{\partial g(m)}{\partial m}\frac{1}{g(m)} dm, \tag{B.1}$$

and, applying this definition to the Fokker-Planck equation (3), the particular effective potential for the Kirman dynamics is found to be

$$V_{\text{eff}}(m) = (h-a)\ln(1-m^2).$$

Note that this effective potential $V_{\text{eff}}(m)$ is not to be confused with the deterministic potential, which is always monostable and can be found by simply integrating with respect to m the deterministic part of equation (5).

Even though in the case with an external periodic forcing it is not possible to write a stationary state probability distribution, we assume that, at any point in time, the decay of the system to a "quasi-stationary state" is faster than the variation of the input signal. Therefore, we keep the previous definition of the effective potential as an approximation to this time dependent case,

$$P(m,t) \approx \mathscr{Z}^{-1} \exp\left(-\frac{V_{\text{eff}}(m,t)}{D}\right).$$

Thereby, applying equation (B.1) to the Fokker-Planck equation (9) leads to the particular functional form

$$V_{\text{eff}}(m,t) = (h_0-a)\ln(1-m^2) - mF\cos(\Omega t)$$

for the forced Kirman dynamics.

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